

SUMMARY OF THE PHD THESIS ENTITLED

Graph Structures from Combinatorial Optimization and Rigidity Theory

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Introduction

The thesis investigates several topics from combinatorial optimization and combinatorial rigidity theory. These topics are the following:

- A tree-composition is a tree-like family that serves to describe the obstacles to k -edge-connected orientability of mixed graphs. *We derive a structural result on tree-compositions* that gives rise to a simple algorithm for computing an obstacle when the orientation does not exist. *We also investigate the application of this result to the orientability of graphs* with several other connectivity prescriptions. With a different method *we give a simpler algorithm to find a rooted $(k, 1)$ -edge-connected orientation* of a graph.
- *We give a generalization of a result of Kamiyama, Katoh and Takizawa [12] on arborescence packings* where a matroid is given on the roots of the arborescences as in the paper of Durand de Gevigney, Nguyen and Szigeti [5].
- We give a negative answer to a question of Graver, Servatius and Servatius [10]. *We prove that there exist circuits of the 2-dimensional rigidity matroid with maximum degree 4 which do not contain any Hamiltonian path nor a path longer than $|V|^\lambda$ for $\lambda > \frac{\log 8}{\log 9} \simeq 0,9464$.* Moreover, we prove a similar statement for every $(k, k+1)$ -sparsity matroid.
- *We extend the rigidity augmentation algorithm of García and Tejel [9] for every (k, ℓ) -sparsity matroid where $\ell \leq \frac{3}{2}k$.* That is, we give a polynomial algorithm to the following augmentation problem for $\ell \leq \frac{3}{2}k$. Given a (k, ℓ) -tight graph $G = (V, E)$, find a graph $H = (V, F)$ with minimum number of edges, such that $G+H-e$ has (k, ℓ) -tightspanning subgraph for every $e \in E$. We also give a polynomial algorithm for the case where $G = (V, E)$ is not (k, ℓ) -tight but has a spanning (k, ℓ) -tight subgraph when $\ell \leq k$ and show that this problem is NP-hard when $\ell = \frac{3}{2}k$.
- *We investigate the edge number of minimal highly node-redundantly rigid and globally rigid graphs.* We give several lower and upper bounds along with some tight examples.
- *We characterize the global rigidity of generic body-hinge and body-bar-and-hinge frameworks.* A surprising point in this result is that it disproves several well-known conjectures on global rigidity in $d \geq 3$ by giving infinitely many counterexamples for Hendrickson's conjecture in every dimension $d \geq 3$.
- Tensegrity frameworks are defined on a set of points in \mathbb{R}^d and consist of bars, cables, and struts, which provide upper and/or lower bounds for the distance between their endpoints. *We prove that it is NP-hard to determine whether every generic realization of the (edge-labeled) graph of a tensegrity framework is rigid on the line.*

Tree-compositions and graphs with connected orientations

Two subsets A and $B \subseteq V$ are **crossing** if $A - B \neq \emptyset$, $B - A \neq \emptyset$, $A \cap B \neq \emptyset$ and $A \cup B \neq V$. A family \mathcal{F} of subsets of V is a **crossing family** if $A \cap B \in \mathcal{F}$ and $A \cup B \in \mathcal{F}$ for any two crossing members $A, B \in \mathcal{F}$. A family \mathcal{F} is called **cross-free** if there are no crossing pairs in \mathcal{F} . Edmonds and Giles proved the following representation of cross-free families:

Lemma 1 (Edmonds and Giles [7]). *For every cross-free family \mathcal{F} on a ground set V , there exists a directed tree $T = (U, A)$ and a map $\varphi : V \rightarrow U$ so that the sets in \mathcal{F} and the edges of T are in a one-to-one correspondence, as follows. For every edge $e \in A$, the corresponding set \mathcal{F} is $\varphi^{-1}(U_e)$ where U_e denotes the component of $T - e$ that contains the head of e .*

A cross-free family is a **tree-composition of Z** if the representing tree $T = (U, A)$ has only two levels $X, Y \subseteq U$ (that is, $X \cup Y = U$, $X \cap Y = \emptyset$ and all edges of T has a tail in X and a head in Y), $\varphi(Z) = Y$ and $\varphi^{-1}(u) \neq \emptyset$ for every $u \in U$.

By a $z\bar{s}$ -set we mean a set containing z and not containing s . Let Z be a non-empty proper subset of a ground set V . A family \mathcal{Z} of subsets of V is **Z -separating** if it contains a $z\bar{s}$ -set for every pair $\{z, s\}$ of elements with $z \in Z$ and $s \in V - Z$. We prove that *the tree-compositions of Z are the minimal cross-free Z -separating families*; and based on this structural result we give an algorithmic proof for the following theorem.

Theorem 2 (Frank, K.). *For a given non-empty proper subset Z of V , a crossing and Z -separating family \mathcal{Z} of subsets of V includes a tree-composition of Z .*

Tree-compositions are useful tools in the investigation of orientability of graphs with connectivity prescriptions. The algorithmic proof of Theorem 2 implies a simple algorithm in these applications to find an obstacle that shows that the orientation does not exist. Here we mention an example. For non-negative integers k and ℓ , a digraph $D = (V, A)$ with a root node $r_0 \in V$ is called **r_0 -rooted (k, ℓ) -edge-connected** if there are k arc-disjoint paths from r_0 to every other node and there are ℓ arc-disjoint paths from every node to r_0 . A rooted $(k, 0)$ -edge-connected digraph is called **rooted k -edge-connected**. The algorithm for Theorem 2 can be used to output an obstacle if a mixed graph does not have a rooted (k, ℓ) -edge-connected orientation.

We also investigate the simpler cases when we are searching for rooted k - and $(k, 1)$ -edge connected orientation of an unoriented graph. In these cases, much simpler algorithms can be given.

Arborescence-packings

An arborescence is a directed tree with a root from which every other node is reachable on a one-way path. For a non-empty set $R \subseteq V$, $B = (V, A')$ is said to be an **R -branching**

if it consists of $|R|$ node-disjoint arborescences whose roots are in R . Let $D = (V, A)$ be a digraph. Then an R -branching is said to be **spanning** if it spans the node set V and it is said to be **maximal** if it spans all the nodes that are reachable from R in D . For non-empty sets $X, Z \subseteq V$, let $Z \mapsto X$ denote that X and Z are *disjoint* and X is reachable from Z , that is, there is a directed path from Z to X . Let $P(X) := X \cup \{v \in V - X : v \mapsto X\}$. Edmonds proved the following famous result on packing of branchings.

Theorem 3 (Edmonds [6]). *In a digraph $D = (V, A)$, let $\mathcal{R} := \{R_1, \dots, R_k\}$ be a family of non-empty subsets of V . Then there are edge-disjoint spanning R_i -branchings for $i = 1, \dots, k$ if and only if*

$$\varrho(X) \geq p_{\mathcal{R}}(X) \quad (1)$$

holds for every $X \subseteq V$ where $\varrho(X)$ denotes the in-degree of X in D and $p_{\mathcal{R}}(X)$ denotes the number of the members of \mathcal{R} disjoint from X .

Kamiyama, Katoh and Takizawa extended the result of Edmonds for the case where not all nodes are reachable from the sets R_i 's.

Theorem 4 (Kamiyama, Katoh and Takizawa [12]). *In a digraph $D = (V, A)$, let $\mathcal{R} := \{R_1, \dots, R_k\}$ be a family of non-empty subsets of V . Then there are k edge-disjoint maximal R_i -branchings in D if and only if*

$$\varrho(X) \geq p'_{\mathcal{R}}(X) \quad (2)$$

holds for every $X \subseteq V$ where $p'_{\mathcal{R}}(X)$ denotes the number of R_i 's for which $R_i \mapsto X$.

A recent result of Durand de Gevigney, Nguyen and Szigeti [5] generalizes Edmonds' results [6] in another direction. We say that the quadruple $(D, \mathcal{M}, \mathbf{S}, \pi)$ is a **matroid-based rooted-digraph** if $D = (V, A)$ is a digraph, \mathcal{M} is a matroid on \mathbf{S} with rank function $r_{\mathcal{M}}$ and $\pi : \mathbf{S} \rightarrow V$ is a (not necessarily injective) map. The map π is called **\mathcal{M} -independent** if $\pi^{-1}(v)$ is independent in \mathcal{M} for every $v \in V$. An **\mathcal{M} -based packing of arborescences** in $(D, \mathcal{M}, \mathbf{S}, \pi)$ is a set $\{T_1, \dots, T_{|\mathbf{S}|}\}$ of pairwise edge-disjoint arborescences in D such that T_i has root $\pi(\mathbf{s}_i)$ for $i = 1, \dots, |\mathbf{S}|$ and the set $\{\mathbf{s}_j \in \mathbf{S} : v \in V(T_j)\}$ forms a base of \mathcal{M} for each $v \in V$. The result of [5] is the following:

Theorem 5 (Durand de Gevigney, Nguyen and Szigeti [5]). *Let $(D, \mathcal{M}, \mathbf{S}, \pi)$ be a matroid-based rooted-digraph. There exists an \mathcal{M} -based packing of arborescences in $(D, \mathcal{M}, \mathbf{S}, \pi)$ if and only if π is \mathcal{M} -independent and*

$$\varrho(X) \geq r_{\mathcal{M}}(\mathbf{S}) - r_{\mathcal{M}}(\pi^{-1}(X)) \quad (3)$$

holds for each $X \subseteq V$.

We give a common generalization of Theorems 4 and 5, as follows. We call a **maximal \mathcal{M} -independent packing of arborescences** a set $\{T_1, \dots, T_{|\mathcal{S}|}\}$ of pairwise edge-disjoint arborescences for which T_i has root $\pi(s_i)$ for $i = 1, \dots, |\mathcal{S}|$, the set $\{s_j \in \mathcal{S} : v \in V(T_j)\}$ is independent in \mathcal{M} and $|\{s_j \in \mathcal{S} : v \in V(T_j)\}| = r_{\mathcal{M}}(\pi^{-1}(P(v)))$.

Theorem 6. *Let $(D, \mathcal{M}, \mathcal{S}, \pi)$ be a matroid-based rooted digraph. There exists a maximal \mathcal{M} -independent packing of arborescences in $(D, \mathcal{M}, \mathcal{S}, \pi)$ if and only if π is \mathcal{M} -independent and*

$$\varrho(X) \geq r_{\mathcal{M}}(\pi^{-1}(P(X))) - r_{\mathcal{M}}(\pi^{-1}(X)) \quad (4)$$

holds for each $X \subseteq V$.

Results on sparse graphs

A graph $G = (V, E)$ is called (k, ℓ) -**sparse** if $i_G(X) \leq k|X| - \ell$ for all $X \subseteq V$ with $|X| \geq 2$, where k and ℓ are integers with $k > 0$ and $\ell < 2k$, (and $i_G(X)$ denotes the number of edges induced by X). A (k, ℓ) -sparse graph is called (k, ℓ) -**tight** if $|E| = k|V| - \ell$. Sparse graphs are important in rigidity theory as many rigidity classes can be characterized using sparse graphs (see Laman's result [13] that characterizes the minimally rigid graphs on the plane with $(2, 3)$ -tight graphs and Tay's result [16, 18] on the rigidity of body-bar frameworks in \mathbb{R}^d). Following the appellations of rigidity theory, we call G (k, ℓ) -**rigid** if G has a (k, ℓ) -tight spanning subgraph. We will call an edge e of G a (k, ℓ) -**redundant edge** if $G - e$ is (k, ℓ) -rigid. A graph G will be called a (k, ℓ) -**redundant graph** if each edge of G is (k, ℓ) -redundant.

It is known that the edge sets of the (k, ℓ) -sparse subgraphs of a given graph form a matroid, called the (k, ℓ) -**sparsity matroid** or **count matroid**. A circuit of this matroid is called an (k, ℓ) -**circuit**. We call a $(2, 3)$ -circuit a **generic circuit**.

Balanced generic circuits without long paths

Graver, B. Servatius and H. Servatius [10] posed the following problem: *Does every generic circuit with nodes of degrees 3 and 4 only, have a decomposition into two Hamiltonian paths?* One can extend this question for $(k, k+1)$ -circuits that has nodes with degrees $2k-1$ and $2k$ only. Such a graph in which the difference between the maximum and minimum degree is at most one is called **balanced**. Note that the smallest (simple) $(k, k+1)$ -circuit is K_{2k} . It is well known that every $(k, k+1)$ -circuit is decomposable into k spanning trees, and balanced $(k, k+1)$ -circuit has nodes with degrees $2k-1$ and $2k$ only and the number of nodes with degree $2k-1$ is exactly $2k$. Hence one may ask if it is decomposable into k Hamiltonian paths.

We give a negative answer for the question. Moreover, we have a stronger result on the length of the longest paths in balanced $(k, k+1)$ -circuits, as follows.

Theorem 7 (K., Péterfalvi). *For all $\lambda > \frac{\log 8}{\log 9}$ and for all $k \geq 2$ there are (infinitely many) balanced $(k, k+1)$ -circuits without any path of length $|V|^\lambda$.*

Augmenting rigidity

We consider the following augmentation problem that we call here the **general (augmentation) problem**.

Problem. *Let k and ℓ be integers with $k \geq 0$ and let $G = (V, E)$ be a loopless (k, ℓ) -rigid graph. Find a graph $H = (V, F)$ on the same node set with minimum number of edges, such that $G + H = (V, E \cup F)$ is (k, ℓ) -redundant.*

We call the special case of this problem, where the input graph G is (k, ℓ) -tight, the **reduced (augmentation) problem**.

As sparsity properties are important in rigidity theory, it is natural to ask how many new edges are needed to make a rigid graph redundantly rigid, that is, the augmented graph remains rigid if we omit an arbitrary edge of it. García and Tejel [9] showed that this is NP-hard for $(2, 3)$ -rigid graphs but can be solved polynomially for minimally rigid graphs, that is, when G is $(2, 3)$ -tight.

We use the idea of García and Tejel [9] to give a polynomial algorithm that solves the reduced problem for $\ell \leq \frac{3}{2}k$ for every k . The algorithm is a greedy-type algorithm with two phases thus it is quite simple. We also show how to use this algorithm to give a polynomial algorithm that solves the general problem for $\ell \leq k$. On the other hand, we extend the NP-hardness result of [9] for the case where k is even and $\ell = \frac{3}{2}k$.

Results on rigid and globally rigid of graphs

A **d -dimensional framework** is a pair (G, p) , where $G = (V, E)$ is a graph and p is a map from V to \mathbb{R}^d . We will also refer to (G, p) as a **realization** of G . (G, p) is a **generic** realization if the elements of the set $\{p(v_i)_j : i = 1, \dots, |V|, j = 1, \dots, d\}$ are algebraically independent over \mathbb{Q} . Two realizations (G, p) and (G, q) of G are **equivalent** if $\|p(u) - p(v)\| = \|q(u) - q(v)\|$ holds for all pairs u, v with $uv \in E$. Frameworks (G, p) and (G, q) are **congruent** if $\|p(u) - p(v)\| = \|q(u) - q(v)\|$ holds for all pairs u, v with $u, v \in V$.

We say that (G, p) is **globally rigid** in \mathbb{R}^d if every d -dimensional framework which is equivalent to (G, p) is also congruent to (G, p) . A framework (G, p) is **rigid** if there exists an $\varepsilon > 0$ such that, if (G, q) is equivalent to (G, p) and $\|p(u) - q(u)\| < \varepsilon$ for all $v \in V$, then (G, q) is congruent to (G, p) .

We say that the graph G is **rigid** in \mathbb{R}^d if every (or equivalently, if some) generic realization of G in \mathbb{R}^d is rigid. We say that a graph G is **globally rigid** in \mathbb{R}^d if every (or equivalently, if some) generic realization of G in \mathbb{R}^d is globally rigid.

Node-redundantly rigid and globally rigid graphs

A graph $G = (V, E)$ is called **k -rigid** in \mathbb{R}^d , or simply **$[k, d]$ -rigid**, if $|V| \geq k + 1$ and, for any $U \subseteq V$ with $|U| \leq k - 1$, the graph $G - U$ is rigid in \mathbb{R}^d . Similarly, one can define **globally $[k, d]$ -rigid graphs** by substituting globally rigid instead of rigid in the definition. Every $[k, d]$ -rigid graph is $[\ell, d]$ -rigid by definition for $1 \leq \ell \leq k$.

G is called **minimally $[k, d]$ -rigid** if it is $[k, d]$ -rigid but $G - e$ fails to be $[k, d]$ -rigid for every $e \in E$. G is said to be **strongly minimally $[k, d]$ -rigid** if it is minimally $[k, d]$ -rigid and there is no (minimally) $[k, d]$ -rigid graph on the same node set with less edges. If G is minimally $[k, d]$ -rigid but not strongly minimally $[k, d]$ -rigid, then it is called **weakly minimally $[k, d]$ -rigid**. Similarly, one can define **weakly/strongly minimally globally $[k, d]$ -rigid** graphs.

We investigate the edge number of minimally $[k, d]$ -rigid and globally $[k, d]$ -rigid graphs. We prove the existence of weakly minimally $[k, d]$ -rigid and globally $[k, d]$ -rigid graphs for every $k \geq 2$ by constructing examples and give several lower and upper bounds, as follows.

Theorem 8 (Kaszanitzky, K.). *Let $G = (V, E)$ be a minimally $[k, d]$ -rigid with $|V| \geq d^2 + d + k$ then*

$$\max \left\{ d|V| - \binom{d+1}{2} + (k-1)d + \max \left\{ 0, \left\lceil k - 1 - \frac{d+1}{2} \right\rceil \right\}, \left\lceil \frac{d+k-1}{2} |V| \right\rceil \right\} \leq |E| \leq (d+k-1)|V| - \binom{d+k}{2}.$$

Moreover, the lower bound is sharp for $k = 2$ for every d and for $k = d = 3$ while the upper bound is sharp for every when $d \geq 2$.

Theorem 9. *Let $G = (V, E)$ be a minimally globally $[k, d]$ -rigid with $|V| \geq d^2 + d + k + 1$ then*

$$\max \left\{ d|V| - \binom{d+1}{2} + (k-1)d + 1 + \max \left\{ 0, \left\lceil k - 1 - \frac{d+1}{2} + \frac{1}{d} \right\rceil \right\}, \left\lceil \frac{d+k}{2} |V| \right\rceil \right\} \leq |E|.$$

This bound is sharp for $d = 2$ when $k = 2$ or 3 (see [14]).

We also form a conjecture that would follow an upper bound of $(d+k)|V| - \binom{d+k+1}{2}$ for the edge number of minimally globally $[k, d]$ -rigid graphs.

Globally rigid body-hinge graphs

One of the important steps towards a possible characterization of global rigidity in higher dimensions is to identify new necessary or sufficient conditions for global rigidity and to characterize global rigidity of special graph classes. We characterize the global rigidity of body-hinge frameworks and show infinitely many examples for graphs that are not globally rigid in dimension $d \geq 3$ but has the following necessary conditions for global rigidity that were also conjectured to be sufficient by Hendrickson.

Theorem 10 (Hendrickson [11]). *Let $G = (V, E)$ be a globally rigid graph in \mathbb{R}^d . Then either G is a complete graph on at most $d + 1$ nodes, or*

- (i) G is $(d + 1)$ -connected, and*
- (ii) $G - e$ is rigid in \mathbb{R}^d for every $e \in E$.*

A d -dimensional **body-hinge framework** is a structural model consisting of rigid bodies and hinges. Each hinge is a $(d - 2)$ -dimensional affine subspace that joins some pair of bodies. The bodies are free to move continuously in \mathbb{R}^d subject to the constraint that the relative motion of any two bodies joined by a hinge is a rotation about the hinge. In the underlying graph of the framework, the nodes correspond to the bodies and the edges correspond to the hinges. We can obtain an equivalent bar-joint framework by replacing each body by a bar-joint realization of a large enough complete graph in such a way that two bodies joined by a hinge share $d - 1$ joints. The graph of such a bar-joint framework is the **body-hinge graph** that arises from the underlying graph of the body-hinge framework.

Body-hinge frameworks (and body-hinge graphs) are extensively studied objects in rigidity theory with various applications. Among others, they can be used to investigate the flexibility of molecules, due to the fact that molecular conformations can be modeled by body-hinge frameworks with certain additional geometric constraints. Tay [17, 19] and Whiteley [20] characterized rigid d -dimensional body-hinge graphs in terms of their underlying graphs.

Theorem 11 (Tay [17, 19] and Whiteley [20]). *Let $H = (V, E)$ be a graph. Then the body-hinge graph that arises from H is rigid in \mathbb{R}^d if and only if there are $\binom{d+1}{2}$ disjoint spanning trees in $\left(\binom{d+1}{2} - 1\right) H$.*

Connelly, Jordán, and Whiteley [3] conjectured a sufficient condition for the global rigidity of body-hinge graphs. We give an affirmative answer to their conjecture. Furthermore, we show that the conjectured sufficient condition is also necessary.

Theorem 12 (Jordán, K., Tanigawa). *Let $H = (V, E)$ be a graph and $d \geq 3$. Then the body-hinge graph that arises from H is globally rigid in \mathbb{R}^d if and only if there are $\binom{d+1}{2}$ disjoint spanning trees in $\left(\binom{d+1}{2} - 1\right) H - e$ for every $e \in E$.*

We also show a similar characterization for globally rigid ‘body-bar-and-hinge’ graphs.

We say that a graph G is an **H-graph** in \mathbb{R}^d if it satisfies the conditions of Theorem 10) but it is not globally rigid in \mathbb{R}^d . For $d = 3$, Connelly [1] showed that the complete bipartite graph $K_{5,5}$ is an H-graph. He presented H-graphs for all $d \geq 3$ as well. These H-graphs are all complete bipartite graphs on $\binom{d+2}{2}$ nodes. Frank and Jiang [8] found two more (bipartite) H-graphs in \mathbb{R}^4 and infinite families of H-graphs in \mathbb{R}^d for $d \geq 5$. Some of their H-graphs in \mathbb{R}^d , $d \geq 5$, contain the complete graph K_{d+1} as a subgraph.

Connelly conjectured that $K_{5,5}$ is the only H-graph in \mathbb{R}^3 [2, 4]. Connelly and Whiteley [4] conjectured that there exist no H-graphs in \mathbb{R}^d containing K_{d+1} as a subgraph. They also

conjectured that the number of H-graphs is finite in \mathbb{R}^d , for all $d \geq 3$. Although the above mentioned examples [8] disproved the latter conjectures for $d \geq 5$, they remained open in the three- and four-dimensional cases. The surprising application of Theorem 12 is that it gives infinitely many H-graphs for every $d \geq 3$ and these graphs contain K_{d+1} as a subgraph.

Strongly rigid tensegrity graphs on the line

A **tensegrity graph** $T = (V; C, S)$ is a 2-edge-labeled graph on the node set $V = \{v_1, v_2, \dots, v_n\}$ whose edge set is partitioned into two sets C and S , called **cables**, and **struts**, respectively. A d -dimensional **tensegrity framework** is a pair (T, p) , where T is a tensegrity graph and p is a map from V to \mathbb{R}^d .

An **infinitesimal motion** of a tensegrity framework (T, p) is an assignment $q : V \rightarrow \mathbb{R}^d$ of infinitesimal velocities to the nodes, such that

$$\begin{aligned} (p_i - p_j)(q_i - q_j) &\leq 0 \quad \text{for all } v_i v_j \in C, \\ (p_i - p_j)(q_i - q_j) &\geq 0 \quad \text{for all } v_i v_j \in S, \end{aligned}$$

where $p(v_i) = p_i$ and $q(v_i) = q_i$ for all $1 \leq i \leq n$. An infinitesimal motion is **trivial** if it can be obtained as the derivative of a rigid congruence of all of \mathbb{R}^d restricted to the nodes of (T, p) . The tensegrity framework (T, p) is **infinitesimally rigid** in \mathbb{R}^d if all of its infinitesimal motions are trivial. It is well-known that the infinitesimal rigidity of tensegrity frameworks is not a generic property: the same tensegrity graph may possess infinitesimally rigid as well as infinitesimally non-rigid generic realizations in any fixed dimension $d \geq 1$. Thus we may define two (different) families of tensegrity graphs in dimension d : we say that a tensegrity graph T is **rigid** in \mathbb{R}^d if it has an infinitesimally rigid generic realization (T, p) in \mathbb{R}^d and **strongly rigid** if all generic realizations (T, p) in \mathbb{R}^d are infinitesimally rigid.

The rigidity of tensegrity graphs on the line can be tested in polynomial time (see [15]) but it is not known whether it can be tested in polynomial time in \mathbb{R}^d for $d \geq 2$. We show that testing strong rigidity is a more difficult problem even on the line.

Theorem 13 (Jackson, Jordán, K.). *It is co-NP-complete to test whether a tensegrity graph is strongly rigid in \mathbb{R}^1 .*

The thesis is based on the following papers

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